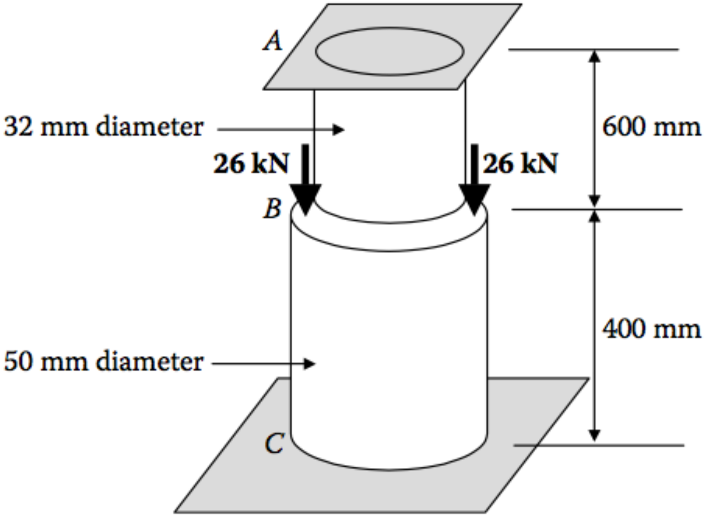
Georg Fantner

Week 4
Over-constrained systems

- Displacement-stiffness method
- Thermal effects
- Saint-Venant's principle
- Stress concentration

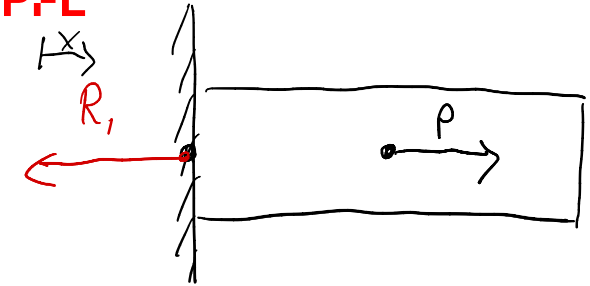


Statically indeterminate bars

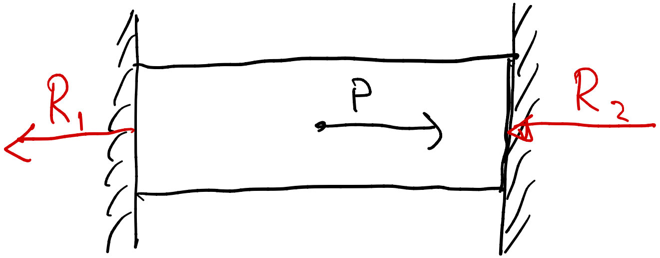
Statically indeterminate system: A system for which the equilibrium equations (expressed in terms of stresses) are insufficient to determine the reactions.

Some of the loads or supports are redundant to maintain equilibrium

There are fewer (non trivial) equations of static equilibrium available than there are unknown reactions.



$$\sum F_x = -R_1 + P = 0 \Rightarrow R_1 = P$$



$$\sum F_x = -R_1 + P - R_2 = 0$$

$$R_1 = P - R_2 \Rightarrow R_2 = ???$$

ADDITIONAL INFORMATION!

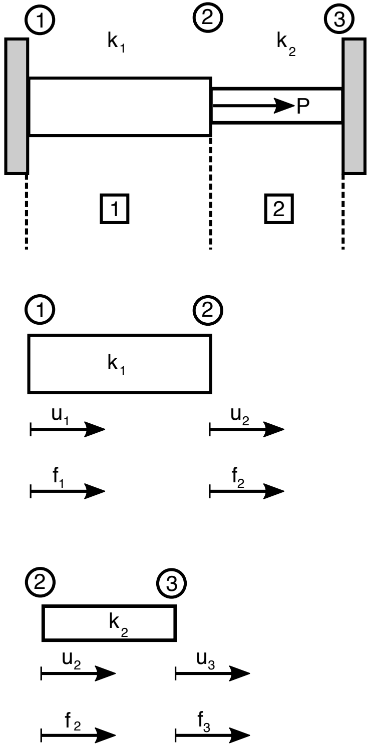
DISPLACEMENT-STIFFNESS METHOD

Approach to solve statically indeterminate systems

Equilibrium: Make use of the fact that equilibrium conditions have to be assured both globally and locally

Constitutive: Hooke's law must be obeyed by all materials of the system.

Kinematic and compatibility: The solution must be compatible with the geometric restraints at the boundary as well as among deformed parts of the body



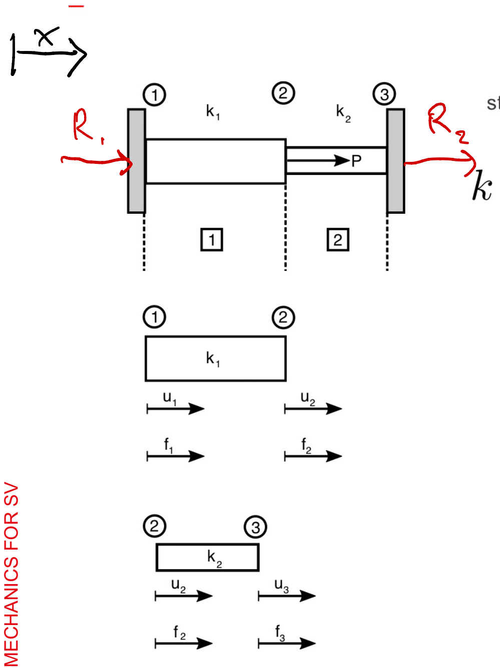
stiffness of bar:

$$k = \frac{AE}{L}$$

The Displacement Stiffness Method

In the displacement method, one postulates a specific <u>displacement</u> and calculates the forces that are required to obtain that displacement. Using *equilibrium* at each node gives us the reaction forces

SEGMENT 1:



stiffness of bar:

$$k = \frac{AE}{L}$$

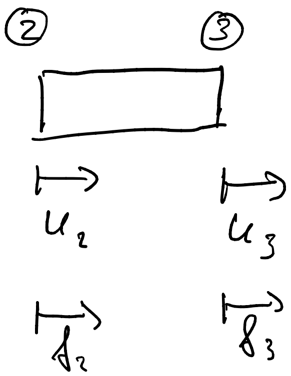
$$f_1 = k_1(u_1 - u_2)$$

$$f_2 = k_1(u_2 - u_1) = -k_1(u_1 - u_2)$$

$$\underline{f_1 = -f_2}$$

$$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \underbrace{\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix}}_{:= k^{\boxed{1}} = \text{STIFFNESS MATRIX OF SEGMENT } \boxed{1}} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

FOR SEGMENT $\boxed{2}$:



$$\begin{Bmatrix} f_2 \\ f_3 \end{Bmatrix} = \underbrace{\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}}_{:= k^{\boxed{2}}} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix}$$

$$k^{\boxed{1}} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} = \begin{pmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$k^{\boxed{2}} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

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$$K = k^{\boxed{1}} + k^{\boxed{2}} = \underbrace{\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}}$$

K := GLOBAL STIFFNESS MATRIX

$$\tilde{F} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

BOUNDARY CONDITIONS: B.C.

$u_1 = 0 \qquad u_3 = 0 \qquad F_2 = P$

$$\begin{Bmatrix} R_1 \\ P \\ R_2 \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \cdot \begin{pmatrix} 0 \\ u_2 \\ 0 \end{pmatrix}$$

$$P = (k_1 + k_2)\, u_2 \quad \Rightarrow \quad u_2 = \frac{P}{k_1 + k_2}$$

$$\begin{Bmatrix} R_1 \\ P \\ R_2 \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{pmatrix} 0 \\ \frac{P}{k_1+k_2} \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{k_1}{k_1+k_2} \\ 1 \\ -\frac{k_2}{k_1+k_2} \end{pmatrix} P$$

The Displacement Stiffness Method

1. Determine the number of redundants and identify the nodes

2. Separate the structure in sections with nodes on each side

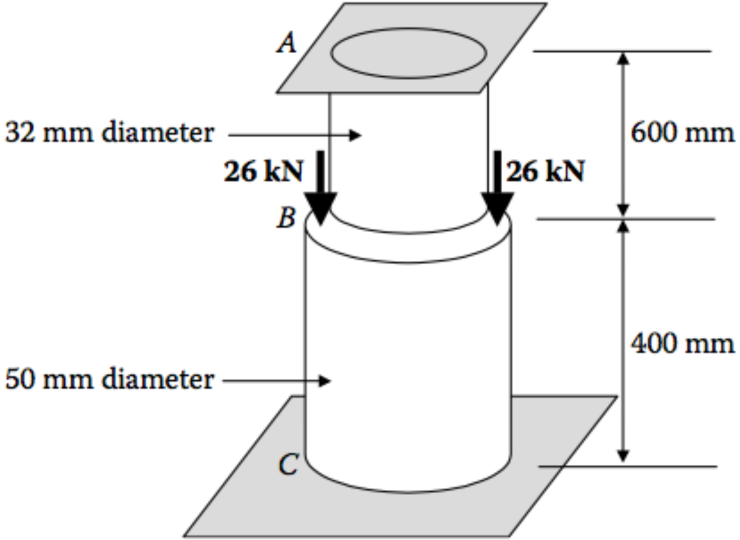
3. Postulate a nodal displacement u_i for each node i and calculate for each node the force that must be acting f_i

4. Calculate the local stiffness matrix for each segment

5. Calculate the global stiffness matrix from the local matrices

6. Enter the boundary conditions and known loads

7. Solve for the remaining unknowns



Example

A polystyrene bar consisting of two cylindrical portions AB and BC is restrained at both ends and supports two 26 kN loads as shown in Fig- ure 2.35. Knowing that E is 3.1 GPa, determine the reactions at A and C

Given: Dimensions of and loading on composite polystyrene bar.

Find: Reactions and normal stresses.

Assume: Hooke’s law applies. Neglect weight of polystyrene cylinders.

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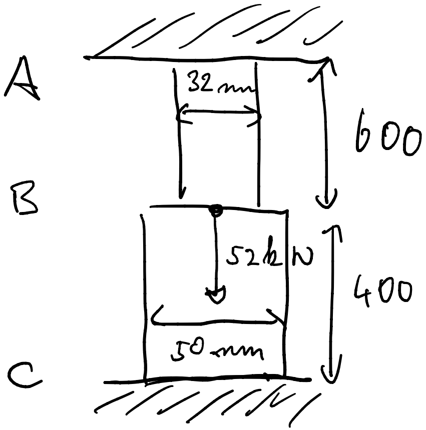
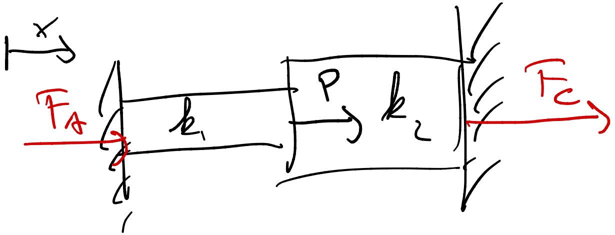




GIVEN: - GEOMETRY: $r_{AB} = 16$ mm $\qquad L_{AB} = 600$ mm

$r_{BC} = 25$ mm $\qquad L_{BC} = 400$ mm

- LOADS: $P = 52$ kN

- $E = 3{,}1$ GPa

ASKED: REACTION FORCES IN A & B

Gov. princ: D. S. METHOD $k = \frac{AE}{L}$

Solution

$$\vec{F} = \begin{Bmatrix} F_A \\ F_B \\ F_C \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{pmatrix} u_A \\ u_B \\ u_C \end{pmatrix}$$

B.C.: $u_A = 0 \quad u_C = 0 \quad F_B = P = 52\ kN$

$$\begin{Bmatrix} F_A \\ P \\ F_C \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{pmatrix} 0 \\ u_B \\ 0 \end{pmatrix}$$

$P = (k_1 + k_2)\, u_B$

$\Rightarrow u_B = \frac{P}{k_1+k_2}$

$$\begin{Bmatrix} F_A \\ P \\ F_C \end{Bmatrix} = \begin{bmatrix} & & \\ & K & \\ & & \end{bmatrix} \begin{pmatrix} 0 \\ \frac{P}{k_1 + k_2} \\ 0 \end{pmatrix} = \begin{Bmatrix} \frac{-k_1}{k_1 + k_2} \\ 1 \\ \frac{-k_2}{k_1 + k_2} \end{Bmatrix} P$$

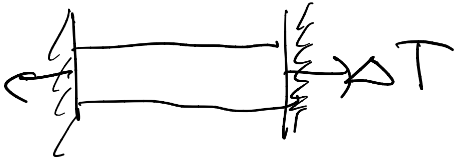
$$k_1 = \frac{A_{AB} E}{L_{AB}} = \frac{\pi r_{AB}^2}{L_{AB}} E$$

$$= \frac{\pi (16 \cdot 10^{-3})^2}{0.6} \cdot 3.1 \cdot 10^9 \text{ Pa} = 4.16 \cdot 10^6 \frac{N}{m}$$

$$k_2 = \frac{A_{BC} E}{L_{BC}} = 1.52 \cdot 10^7 \frac{N}{m}$$

$$\boxed{\begin{aligned} F_A &= \frac{-4.16 \cdot 10^6}{4.16 \cdot 10^6 + 1.52 \cdot 10^7} \, 52 \text{ kN} = -11.1 \text{ kN} \\ F_C &= \frac{-15.2}{4.16 + 15.2} \cdot 52 \text{ kN} = -40.9 \text{ kN} \end{aligned}}$$

Thermal effects



Materials expand with an increase in temperature: this is called a thermal strain

$$\varepsilon_T = \alpha(T - T_0) = \alpha\Delta T$$

α is the coefficient of (linear) thermal expansion. It has a dimension of (mm/mm)/°C or $°C^{-1}$

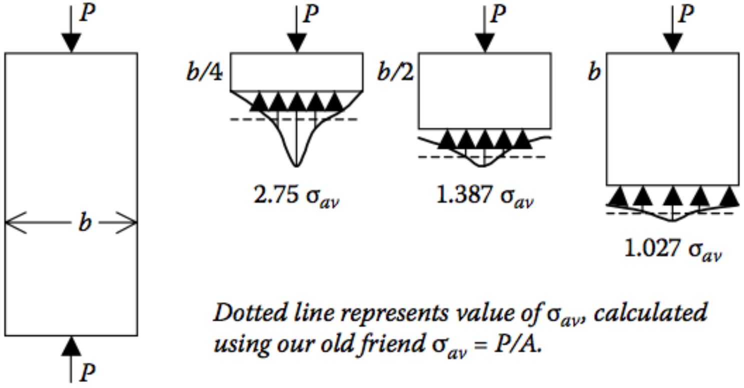
If the material body is constrained, the thermal strain will result in a thermal stress:

$$\sigma_T = E\alpha(\Delta T)$$

Thermal stresses and strains can be super-positioned with normal stresses and strains

"This project calls for real concentration.
Are you still able to monotask?"

Stress concentration



Dotted line represents value of σ_{av}, calculated using our old friend $\sigma_{av} = P/A$.

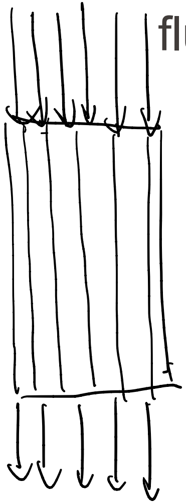
Saint-Venant's principle

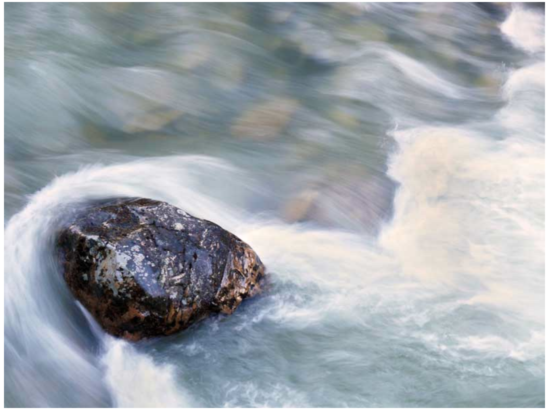
- Forces in reality are rarely distributed uniformly across their surface of action
- *Saint-Venant's Principle* states that the manner of force application only plays a role near the point of force application
- For a bar loaded with a point load we can show that the normal stresses are nearly uniform on a surface whose distance from the applied force is the same as the width of the body.

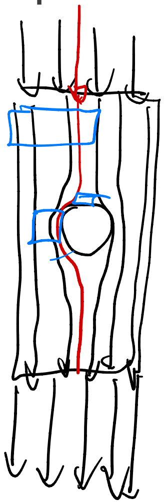
Stress concentration

The analogy to fluid flow rates

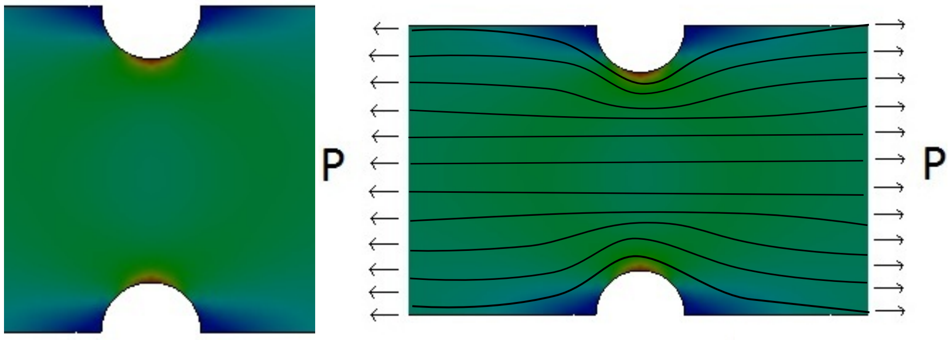
- Stress concentration can be thought of in similar terms as the flow speed in a fluid when it get’s to an area with reduced cross section

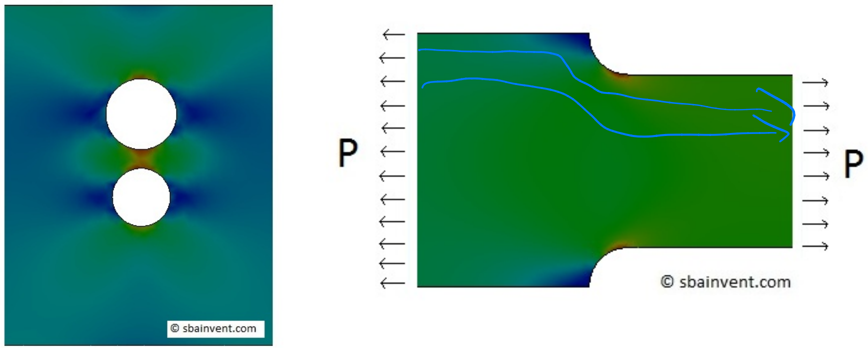






  - Stone with water flowing around is analogue to a bar with a hole in it
  - water flowing between two stones is analog to a bar with two notches.



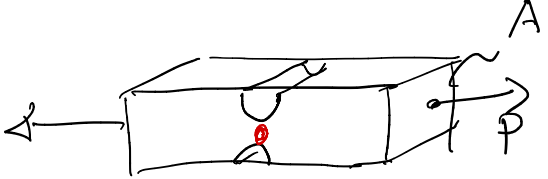


Stress concentration

The analogy to flow

- Picture that a fluid has to flow through your structure.
- In the areas where you have an increase in fluid flow velocity you have a positive stress concentration (local increase in stress).
- In areas where you have a decreased fluid flow velocity you have a negative stress concentration (a local decrease in stress).

Stress concentration

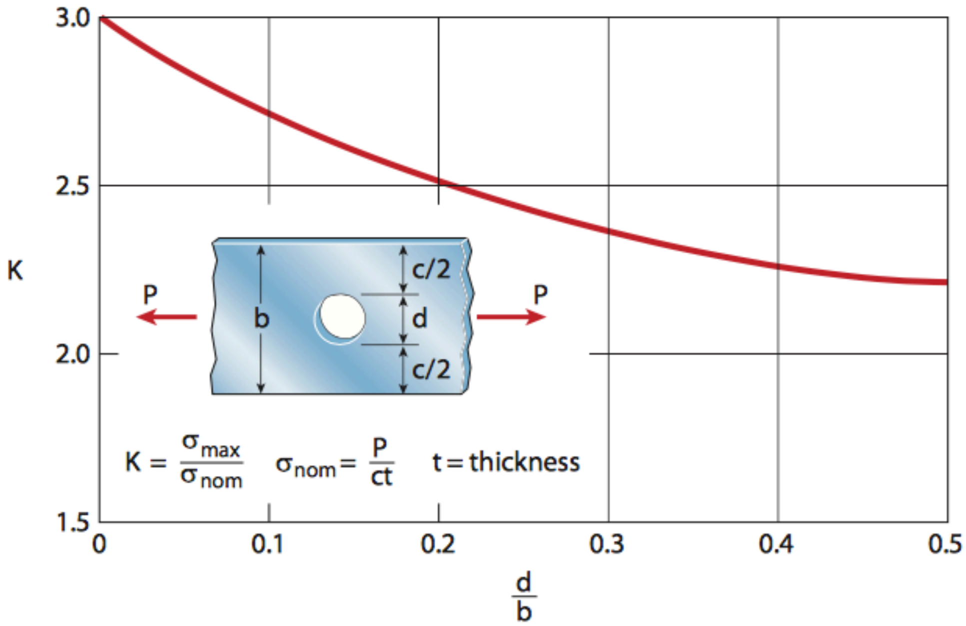


- From Saint-Venant's principle we also know that the maximum stress in a structure and its relation to the average stress is a function of geometry.

$$\boxed{\sigma_{max} = K\sigma_{ave} = K\frac{P}{A}}$$

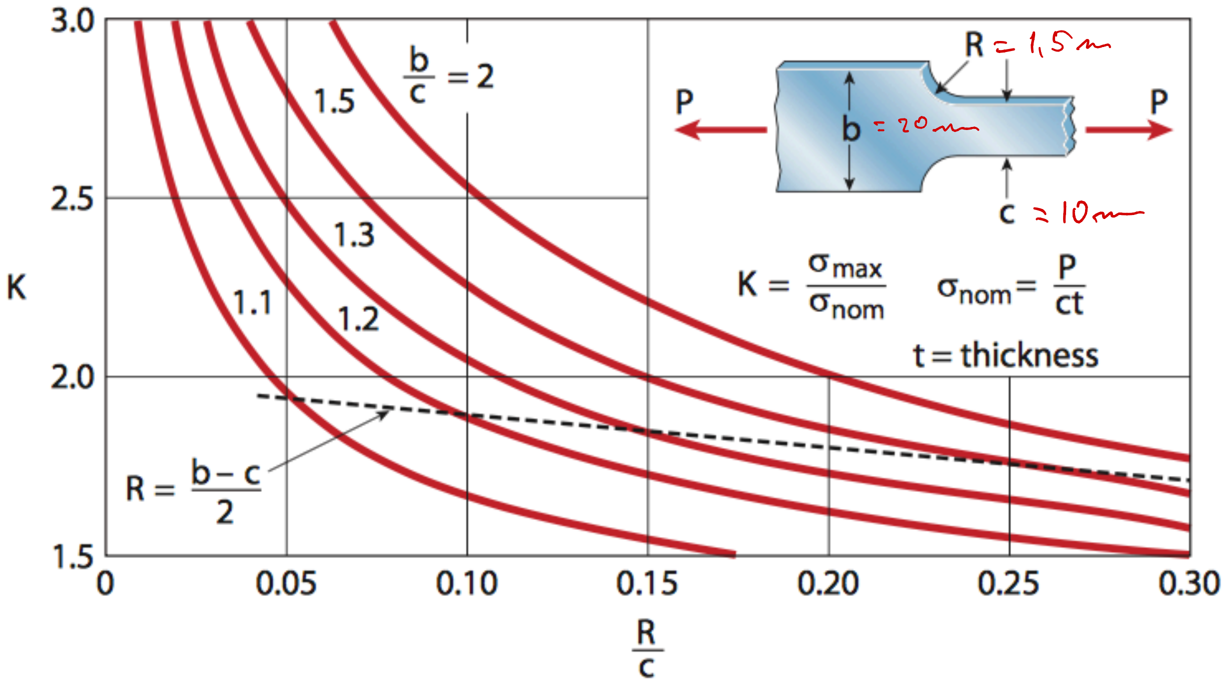
- K is the stress concentration factor and can be determined experimentally or numerically
- K can be looked up in graphs or tables for different geometries (be careful how the σ_{ave} is defined for that graph or table)

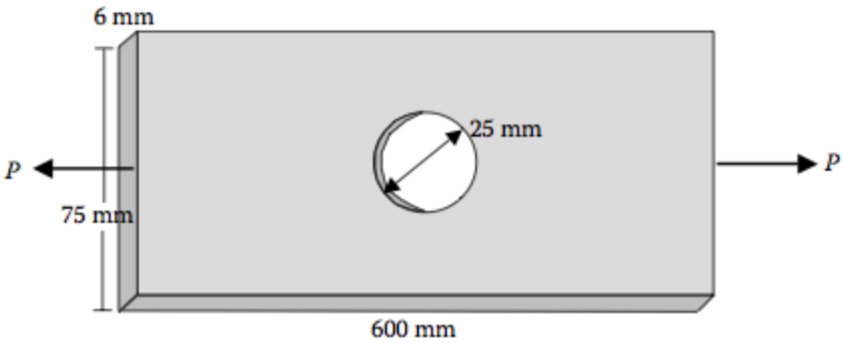
Stress concentration around a hole

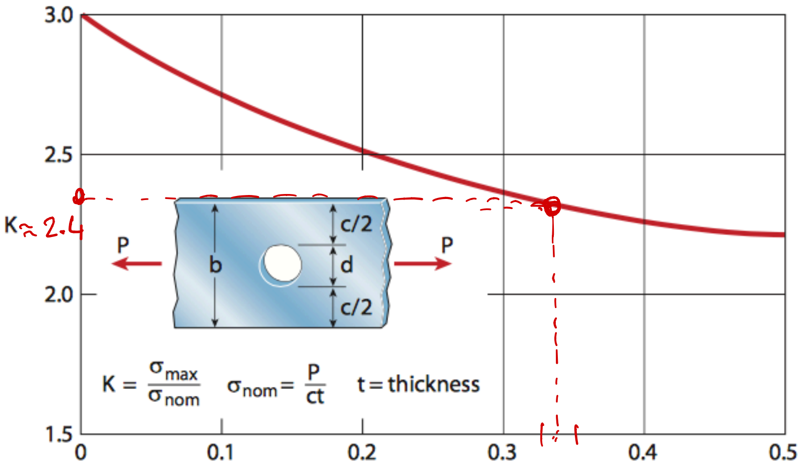


$$\sigma_{max} = K \cdot \sigma_{nom} = K \cdot \frac{P}{c \cdot t}$$

Stress concentration at a change in cross-section







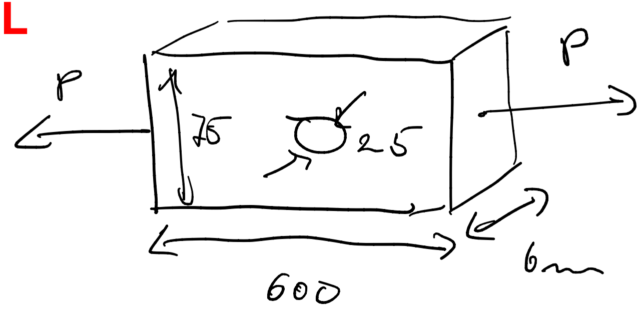
$$\frac{d}{b} = \frac{25}{75} \approx \frac{1}{3}$$

Example 2.9

Stress concentration

A 6 mm × 75 mm plate, 600 mm long, has a circular hole of 25 mm diameter located at its center. Find the axial tensile force that can be applied to this plate in the longitudinal direction without exceeding an allowable stress of 220 MPa. How does the presence of the hole affect the strength of the plate?

Given: Dimensions of plate, limiting normal stress.
Find: Allowable axial load that can be applied to plate. Assume: Hole is only feature that causes a stress concentration.



given: - Geometry

- $\sigma_{MAX} = 220\ MPa$

Asked: P_{MAX}

Gov. princ. $\sigma_{MAX} = K \cdot \sigma_{NOM} = K \dfrac{P}{ct}$

Solution: $k \approx 2.4$

$$\sigma_{MAX} = K \frac{P_{MAX}}{c\,t} \Rightarrow P_{MAX} = \frac{\sigma_{MAX} \cdot c \cdot t}{K} = \frac{220 \cdot 10^{6} \cdot 5 \cdot 10^{-2} \cdot 6 \cdot 10^{-3}}{2.4} \approx \underline{\underline{27.5\ kN}}$$

Another use of treatment of bodies like springs: Strain Energy.

- From the axially loaded bar we have seen the communality of the load extension curve to the basic force extension curve of a spring
- We can now also apply this communality to energy stored in a stretched spring or bar

	Spring	Axially loaded bar
Hooke's Law	$F = k \cdot \Delta x$	$P = \frac{AE}{L} \cdot \delta$
Spring constant	k	$k = \frac{AE}{L}$

Strain Energy in one Dimension

- We know from Hooke's law that a solid material reacts to a load in a similar way as a linear spring. The energy stored by a compressed spring is:

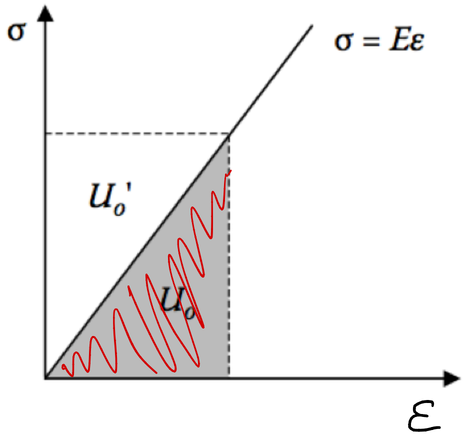
$$U_{spring} = \int_0^x F_s dx = \int_0^x kx dx = \frac{1}{2} kx^2.$$

- In analogy, the strain energy stored in an elastic solid (strain energy per unit volume) is then:

$$U_0 = \int_0^\varepsilon \sigma d\varepsilon = \int_0^\varepsilon E\varepsilon d\varepsilon = \frac{1}{2} E\varepsilon^2.$$

Strain energy in one dimension

- Area under the stress-strain curve: *strain energy density (U_0)*
- Area "above" the stress-strain curve: *complementary energy density (U_0')*

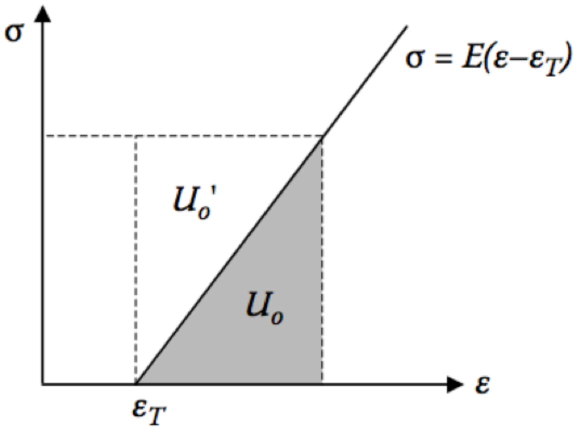


Strain Energy in one Dimension: Thermal strain energy

- Thermal strains offset the stress-strain curve along the strain axis

$$U_0 = \int_{\varepsilon_T}^{\varepsilon} \sigma d\varepsilon = \int_{\varepsilon_T}^{\varepsilon} E(\varepsilon - \varepsilon_T) d\varepsilon = \frac{1}{2} E(\varepsilon - \varepsilon_T)^2$$

$$\boxed{U_0 = \frac{1}{2} E\varepsilon^2 - E\varepsilon(\alpha T) + \frac{1}{2} E(\alpha T)^2}$$



The three governing physical principles for structural mechanics

Equilibrium: relates external forces, reactions, internal forces and stresses

Newton 2: relates external forces to reactions

Method of sections: relates external forces & reactions to internal forces

Newton 2 & method of sections: relate internal forces to stresses

Constitutive laws: relate stresses to strains

Compatibility considerations: relates strains to displacements or deflections (i.e. kinematics)

Review: the three equations

In structural mechanics, we (always) rely on these 3 equations:

Equilibrium equation: ensures that all forces are in equilibrium

$$\frac{dN(x)}{dx} + \sum_i P_i \delta(x - x_i) + B_x A(x) = 0$$

Constitutive equation: Relates two quantities with materials specific properties.

$$E = \frac{\sigma}{\varepsilon}$$

Kinematic Equation: relates strain (ε) to displacement (u):

$$\varepsilon(x) = \frac{du(x)}{dx}$$

Review: stress strain in 1D

Learning objectives of chapter 2

Bars	Calculate normal and shear stresses and strains in bars ("bar in tension formula")
Displacement	Understand the concept of the displacement u(x) and how we derived the strain from this
Hooke	Know Hooke's law and how and when it applies
Sections	Understand the method of sections and be able to solve structures with trusses in 2D
Forces	Know what kind of forces can act on structures (interneal, external, distributed, body, etc…)
Supposition	Understand and apply *superposition principle*

Review: stress strain in 1D

Learning objectives of chapter 2

3 equations	Know and apply *the kinematic, constitutive*, and *equilibrium* equations
Indeterminate Systems	Solve statically indeterminate systems with the *displacement stiffness* method
Thermal	Know and solve problems involving thermal stresses
St.Venant	Understand *Saint Venant's* principle and explain why it is important
Concentration	Know and apply stress concentrations
Energy	Calculate strain Energy